

SEIFERT G_m -BUNDLES

JÁNOS KOLLÁR

Seifert fibered 3-manifolds were introduced and studied in [Sei32]. (See [ST80] for an English translation.) Roughly speaking, these are 3-manifolds M which admit a differentiable map $f : M \rightarrow F$ to a surface F such that every fiber is a circle. Higher dimensional Seifert fibered manifolds were investigated in [OW75]. The authors observed that in many cases of interest Seifert fibered manifolds correspond to holomorphic Seifert \mathbb{C}^* -bundles, and started to develop a general theory of holomorphic Seifert G -bundles for any complex Lie group G .

Definition 1. Let X be a normal variety (or algebraic space) over a field k and G a (reduced) algebraic group over k . A *Seifert G -bundle* over X is a normal variety (or algebraic space) Y together with a morphism $f : Y \rightarrow X$ and a G -action on Y satisfying the following two conditions.

- (1) f is affine and G -equivariant (with respect to the trivial action on X).
- (2) For every $x \in X$, the G -action on the reduced fiber $Y_x := \text{red } f^{-1}(x)$ $G \times Y_x \rightarrow Y_x$ is G -equivariantly isomorphic to the natural (left) G -action on G/I_x for some finite subgroup $I_x \subset G$.

Let $G \times Y \rightarrow Y$ be a proper G -action on a normal variety (or algebraic space). The geometric quotient Y/G exists as an algebraic space (cf. [Kol97, KM97]) and $f : Y \rightarrow Y/G$ is a Seifert G -bundle. As with many quotient problems, even if Y is a quasi projective variety, X need not be quasi projective. (Many such examples are given in [Cox95].) Thus it is natural to work with algebraic spaces. This is, however, a purely technical point, and makes no difference for our purposes. In the questions motivating this work, X is always projective.

One can thus view the theory of Seifert G -bundles as a special chapter of the study of algebraic group actions. The emphasis is, however, quite different. [OW75] studied Seifert S^1 -bundles over \mathbb{CP}^n in order to construct exotic spheres, extending and reformulating earlier results of [Bri66].

In a series of papers, Boyer and Galicki developed a method to construct Einstein metrics on links of certain weighted homogeneous singularities [BG01, BGN02, BGN03a, BGN03b, BG03, BGK03]. These links are Seifert S^1 -bundles over the corresponding projective hypersurfaces in weighted projective spaces. Seifert S^1 -bundles over smooth rational surfaces were used to construct Einstein metrics on connected sums of $S^2 \times S^3$ [Kol04a], but further results need the study of Seifert S^1 -bundles over singular surfaces as well [Kol04b].

The main aim of this paper is to provide the necessary foundations on Seifert \mathbb{C}^* -bundles to continue work in this direction, but Seifert G_m -bundles are also interesting from other points of view, see [Dol75, Pin77, Dem88, FZ03].

Ultimately one is mostly interested in Seifert S^1 -bundles $f : M \rightarrow X$ where M is a manifold and X an orbifold, but the general theory is the same for any normal variety X .

A complete classification of holomorphic Seifert \mathbb{C}^* -bundles over smooth projective varieties X such that $H_1(X, \mathbb{Z})$ is torsion free is given in [OW75]. In section 1 this is extended to any normal variety in any characteristic (7). The local structure of Seifert G_m -bundles is studied in section 2. For applications a key point is the smoothness criterion given in (29). Section 3 is devoted to investigating the relationship between holomorphic and topological Seifert \mathbb{C}^* -bundles. The most interesting Seifert G_m -bundles, those with a smooth total space, are studied in section 4. Finally Section 5 contains some information about the topology of Seifert \mathbb{C}^* -bundles. The computation of the cohomology groups $H^i(Y, \mathbb{Q})$ is easy, but the much more interesting torsion in $H^i(Y, \mathbb{Z})$ remains mostly unexplored.

1. CLASSIFICATION OF SEIFERT G_m -BUNDLES

Notation 2. G_m denotes the multiplicative group $GL(1)$. As a scheme, it is $\mathrm{Spec}_k k[t, t^{-1}]$. The M th roots of unity form the subgroup scheme

$$\mu_M := \mathrm{Spec}_k k[t, t^{-1}]/(t^M - 1).$$

These are all the subgroup schemes of G_m . μ_M is nonreduced when $\mathrm{char} k$ divides M .

Every linear representation $\rho : G_m \rightarrow GL(W)$ is completely reducible, and the same holds for any subgroup $G \subset G_m$ (see, for instance, [DG70, I.4.7.3]). The set of vectors $\{v : \rho(\lambda)(v) = \lambda^i v\}$ is called the λ^i -*eigenspace*. We use this terminology also for μ_M -actions. In this case i is determined modulo M .

If a group G acts on a scheme X via $\rho : G \rightarrow \mathrm{Aut}(X)$, we get an action on rational functions on X given by $f \mapsto f \circ \rho(g^{-1})$. (The inverse is needed mostly for noncommutative groups only.)

Thus if G_m acts on itself by multiplication, we get an induced action on $k[t, t^{-1}]$ where $\lambda \in G_m(\bar{k})$ acts as $t^i \mapsto \lambda^{-i} t^i$. Thus t^i spans the λ^{-i} -eigenspace.

A G_m -action on a k -algebra A is equivalent to a \mathbb{Z} grading $A = \sum_{i \in \mathbb{Z}} A_i$ where A_i is the λ^{-i} -eigenspace.

The natural G_m -action on G_m/μ_M corresponds to the algebra $\sum_{i \in M\mathbb{Z}} k \cong k[t^M, t^{-M}]$.

Definition 3. Let $f : Y \rightarrow X$ be a Seifert G_m -bundle. For every $x \in X$, the G_m -action on the reduced fiber $Y_x := \mathrm{red} f^{-1}(x)$ is isomorphic to the natural G_m -action on $G_m/\mu_{m(x)}$ for some $m(x)$, called the *multiplicity* of the fiber over x . We always assume that $m(x) = 1$ for all x in some dense open subset of X . That is, we assume that the G_m action is effective.

Definition 4. Let X be a normal variety. A *rank 1 reflexive sheaf* is a coherent sheaf L such that L is locally free of rank 1 over a dense open set and the natural map from L to its double dual $L \rightarrow (L^*)^*$ is an isomorphism.

Rank 1 reflexive sheaves on X form a group, called the *class group* of X , denoted by $\mathrm{Cl}(X)$. The group operation is given by the double dual of the tensor product $(L \otimes M)^{**}$. For tensor powers we use the notation

$$L^{[k]} := (L^{\otimes k})^{**}.$$

Let D be a *Weil divisor* on X . Then $\mathcal{O}_X(D)$ is a rank 1 reflexive sheaf and we obtain an isomorphism

$$\mathrm{Cl}(X) \cong \mathrm{Weil}(X)/(\text{linear equivalence}).$$

For a real number s its *round down* (or integral part) is denoted by $\lfloor s \rfloor$.

Our first aim is to classify Seifert G_m -bundles over X in terms of more familiar objects on X .

Definition 5. (cf. [Pin77, Dem88]) Let X be a normal variety, L a rank 1 reflexive sheaf on X , D_i distinct irreducible divisors and $0 < s_i < 1$ rational numbers. Define

$$\begin{aligned} S(L, \sum s_i D_i) &:= \sum_{j \in \mathbb{Z}} L^{[j]}(\sum_i \lfloor j s_i \rfloor D_i) \quad \text{and} \\ Y(L, \sum s_i D_i) &:= \text{Spec}_X S(L, \sum s_i D_i). \end{aligned}$$

There is a natural G_m -action on $S(L, \sum s_i D_i)$ where $L^{[j]}(\sum_i \lfloor j s_i \rfloor D_i)$ is the λ^j eigensubsheaf. (See (6) for choosing λ^j instead of λ^{-j} .) This induces a G_m -action on $Y(L, \sum s_i D_i)$.

When using the above notation we always assume that the multiplicative structure of the algebra $S(L, \sum s_i D_i)$ is given by the tensor product

$$L^{[n]}(\sum_i \lfloor n s_i \rfloor D_i) \otimes L^{[m]}(\sum_i \lfloor m s_i \rfloor D_i) \rightarrow L^{[n+m]}(\sum_i \lfloor (n+m) s_i \rfloor D_i).$$

(This map exists since $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$ for any a, b .)

Remark 6. [Sign choices]

Let $f : L \rightarrow X$ be a line bundle with zero section $X \subset L$. One usually identifies L with its sheaf of sections, temporarily denoted by \mathcal{L} . There is a natural G_m -action on $Y := L \setminus X$ coming from the standard $GL(1)$ action on the fibers of f .

The push forward of the structure sheaf is $f_* \mathcal{O}_L = \sum_{i \geq 0} \mathcal{L}^{-i}$ and \mathcal{L}^{-i} is the λ^{-i} -eigensubsheaf. Thus

$$f_* \mathcal{O}_Y = \sum_{i \in \mathbb{Z}} \mathcal{L}^i,$$

and L is naturally identified with \mathcal{L} , which is the λ^1 -eigensubsheaf.

Special cases of the classification of Seifert G_m -bundles can be found in the papers [Dol75, Pin77, Dem88, FZ03]. The proof of the general case is essentially the same.

Theorem 7. *Let X be a normal variety (or algebraic space).*

- (1) *Every Seifert G_m -bundle $f : Y \rightarrow X$ can be uniquely written as $Y \cong Y(L, \sum s_i D_i)$ for some L and $\sum s_i D_i$ as in (5).*
- (2) *$f : Y(L, \sum s_i D_i) \rightarrow X$ is a Seifert G_m -bundle iff $L^{[M]}(\sum_i (M s_i) D_i)$ is locally free for some $M > 0$ and $M s_i$ is an integer for every i .*
- (3) *The multiplicity of the Seifert fiber Y_x is the smallest natural number $M = m(x)$ such that $L^{[M]}(\sum_i (M s_i) D_i)$ is locally free at x and $M s_i$ is an integer for every i such that $x \in D_i$.*

Remark 8. (1) One can easily see that the proof also works for normal analytic spaces. Thus we obtain that if X is a normal and proper variety over \mathbb{C} then one can naturally identify algebraic and analytic Seifert \mathbb{C}^* -bundles over X .

(2) The result suggests that codimension 1 fixed points of the G_m -action are encoded in the divisorial part $\sum s_i D_i$ while information about higher codimension fixed point sets is carried by the sheaf part L . This is, however, probably the wrong interpretation.

From the stacky point of view of (21) one can identify the pair $(L, \sum s_i D_i)$ with a single rank one reflexive sheaf on the quotient stack Y/G_m .

(3) Without the local freeness assumption in (7.2), the finite generation of the algebra $S(L, \sum s_i D_i)$ is a very subtle question, closely intertwined with the existence of flips, cf. [KM98, Sec.6.1].

(4) The Theorem implies that all Seifert G_m -bundles over X form a group isomorphic to

$$\mathrm{Weil}(X)_{\mathbb{Q}} \times \mathrm{Cl}(X) / \langle (D, \mathcal{O}(-D)) : D \in \mathrm{Weil}(X) \rangle,$$

but I did not find the group structure useful.

Definition 9. [OW75, Vis89] Let $Y \cong Y(L, \sum s_i D_i) \rightarrow X$ be a Seifert G_m -bundle over X . Then $L^{[M]}(\sum_i (M s_i) D_i)$ is locally free and so it is an element of $\mathrm{Pic}(X)$. We can formally divide this by M and get an element, called the *Chern class* of Y/X

$$\begin{aligned} c_1(Y/X) &:= \frac{1}{M} L^{[M]}(\sum_i (M s_i) D_i) \in \mathrm{Pic}(X)_{\mathbb{Q}} \\ &= c_1(L) + \sum_i s_i [D_i]. \end{aligned}$$

If we are over \mathbb{C} , we also get a topological Chern class

$$c_1(Y/X) := \frac{1}{M} c_1(L^{[M]}(\sum_i (M s_i) D_i)) \in H^2(X, \mathbb{Q}).$$

There is very little chance of confusion by using the same notation.

Note that although $c_1(Y/X)$ is a rational class, $M \cdot c_1(Y/X)$ is a well defined class in $H^2(X, \mathbb{Z})$ or in $\mathrm{Pic}(X)$. More generally, if $j : U \hookrightarrow X$ is an open set and $m(U) := \mathrm{lcm}\{m(x) : x \in U\}$ then there is a well defined class

$$m(U) \cdot c_1(Y/X) = c_1(j^*(L^{[m(U)]}(\sum_i (m(U) s_i) D_i))) \in H^2(U, \mathbb{Z})$$

or in $\mathrm{Pic}(U)$.

Similarly, if $s(U) := \mathrm{lcm}\{(\text{denominator of } s_i) : D_i \cap U \neq \emptyset\}$ then there is a well defined class $s(U) \cdot c_1(Y/X) \in \mathrm{Cl}(U)$.

10 (Proof of (7). Let $f : Y \rightarrow X$ be a Seifert G_m -bundle. Since $f : Y \rightarrow X$ is affine, $f_* \mathcal{O}_Y$ is a quasicoherent sheaf with a G_m -action. Thus it decomposes as a sum of G_m -eigensubsheaves

$$f_* \mathcal{O}_Y = \sum_{j \in \mathbb{Z}} L_j, \quad (10.1)$$

where L_j is the λ^j eigensubsheaf, with multiplication maps $m_{ab} : L_a \otimes L_b \rightarrow L_{a+b}$.

Pick any point $x \in X$. By assumption $Y_x \cong G_m / \mu_{m(x)}$, thus $t^{-m(x)}$ on G_m descends to an invertible function h_x on Y_x which is a G_m -eigenfunction with eigencharacter $m(x)$. There is an affine neighborhood $x \in U \subset X$ such that h_x lifts to an invertible function h_U on $f^{-1}(U)$ which is a G_m -eigenfunction with eigencharacter $m(x)$. This h_U is a generator of $L_{m(x)}$ on U and h_U^s is a generator of $L_{m(x)}^{\otimes s}$ on U . Thus for $M = m(X) := \mathrm{lcm}\{m(x) : x \in X\}$, L_M is locally free on X and the multiplication maps $L_M^{\otimes s} \rightarrow L_{sM}$ are isomorphisms for every $s \in \mathbb{Z}$.

Taking M th power gives a map $L_1^{\otimes M} \rightarrow L_M$, hence $L_M \cong L_1^{[M]}(\sum d_i D_i)$ where the D_i are distinct irreducible divisors and $d_i > 0$.

For $j > 0$, the j th power map $L_1^{\otimes j} \rightarrow L_j$ shows that $L_j \subset (L_j)^{**} = L_1^{[j]}(\sum d_{ij} D_i)$ for some d_{ik} (where a priori we may have divisors D_i that do not appear in the expression for L_M above) and the M th power map $L_j^{\otimes M} \rightarrow L_{jM} \cong L_1^{[jM]}(\sum j d_i D_i)$ shows that $\sum d_{ij} D_i \leq \sum \lfloor j d_i / M \rfloor D_i$.

This also holds for $j < 0$ as shown by the isomorphisms $L_j \cong L_{j+M} \otimes L_M^{-1}$.

Thus $f_* \mathcal{O}_Y = \sum_{j \in \mathbb{Z}} L_j$ is a subalgebra of $S(L, \sum (d_i/M) D_i)$ and the two agree in all degrees divisible by M . Therefore we get that $Y(L, \sum (d_i/M) D_i) \rightarrow Y$ is

birational and finite, hence normalization. We have assumed that Y is normal, thus $Y(L, \sum (d_i/M)D_i) \cong Y$. Since $L_1 = L_1^{[1]}(\sum \lfloor d_i/M \rfloor D_i)$, we conclude that $s_i := d_i/M < 1$. This proves (1).

Conversely, assume that $L^{[M]}(\sum_i (Ms_i)D_i)$ is locally free for some $M > 0$ and Ms_i is an integer for every i . We need to prove that $S(L, \sum s_i D_i)$ is a Seifert G_m -bundle. The question is local, so let us fix a point $x \in X$ and let M be the smallest positive number such that $L^{[M]}(\sum_i (Ms_i)D_i)$ is locally free at x and Ms_i is an integer for every i such that $x \in D_i$.

The fiber of $Y(L, \sum s_i D_i) \rightarrow X$ over x is the spectrum of

$$S(L, \sum_i s_i D_i) \otimes k(x) \cong \sum_j L^{[j]}(\sum_i (js_i)D_i) \otimes k(x),$$

where $k(x)$ is the residue field of x .

By (11), the summands corresponding to those j such that $L^{[j]}(\sum_i (js_i)D_i)$ is not locally free are nilpotent, and also those summands where Ms_i is not an integer for every i such that $x \in D_i$. Thus

$$\begin{aligned} Y_x &= \text{Spec}_{k(x)} \left(\sum_j L^{[j]}(\sum_i (js_i)D_i) \otimes k(x) / (\text{nilpotents}) \right) \\ &= \text{Spec}_{k(x)} \sum_{j \in M\mathbb{Z}} k(x) \cong G_m / \mu_M. \end{aligned}$$

This proves parts (2) and (3). \square

Lemma 11. *Let L, M be rank 1 torsion free sheaves and assume that there is a surjective map $h : L \otimes M \rightarrow \mathcal{O}_X$. Then L, M are both locally free.*

Proof. Pick $x \in X$. By assumption there is an affine neighborhood $x \in U$ and sections $\alpha \in H^0(U, L), \beta \in H^0(U, M)$ such that $h(\alpha \otimes \beta)$ is invertible.

Let $\gamma \in H^0(U, L)$ be arbitrary. Then $h(\gamma \otimes \beta) = f \cdot h(\alpha \otimes \beta)$ for some $f \in \mathcal{O}_U$, thus $h((\gamma - f\alpha) \otimes \beta) = 0$. Thus $\gamma - f\alpha$ is zero on the open set where M is locally free, hence it is zero since L is torsion free. Thus α generates $L|_U$ and so L is locally free. \square

Example 12 (Seifert G_m -bundles over curves). Given relatively prime natural numbers $0 < b < c$, let $0 < e < c$ be the unique solution of $be \equiv 1 \pmod{c}$.

Consider $k[s, s^{-1}, u]$ with a G_m -action $s \mapsto \lambda^{-c}s, u \mapsto \lambda^e u$. The G_m -invariants form the polynomial ring $k[s^e u^c]$.

Set $X := \text{Spec}_k[k[s^e u^c]]$ and $Y := \text{Spec}_k k[s, s^{-1}, u] \rightarrow X$. This is a Seifert G_m -bundle with $L_c = s^{-1}k[s^e u^c]$ and $L_1 = u^b s^{(be-1)/c} k[s^e u^c]$. Therefore

$$L_1^{\otimes c} = (u^b s^{(be-1)/c})^c k[s^e u^c] = (s^e u^c)^b L_c.$$

Thus, over the line X we get the Seifert G_m -bundle $Y \cong Y(\mathcal{O}_X, \frac{b}{c}(\text{origin}))$.

The local generating section of L_1 is $h = u^b s^{(be-1)/c}$, thus, as a function on Y , h vanishes along the central fiber with multiplicity b .

Étale locally, this describes Seifert G_m -bundles over curves. More generally, by replacing k with the quotient field of a divisor $D \subset X$, this construction describes the behaviour of an arbitrary Seifert G_m -bundle generically (étale locally) along divisors.

13 (Maps between Seifert G_m -bundles). Let $Y = \text{Spec}_X \sum_{j \in \mathbb{Z}} L_j$ be a Seifert G_m -bundle as in (10.1). For any natural number M , $Y_M := \text{Spec}_X \sum_{j \in M\mathbb{Z}} L_j$ is also

a Seifert G_m -bundle, which can also be viewed as Y/μ_M , the quotient of Y by the action of $\mu_M \subset G_m$. Note that $c_1((Y/\mu_M)/X) = Mc_1(Y/X)$.

The case when $M = m(X)$ is especially interesting, as $\text{Spec}_X \sum_{j \in m(X)\mathbb{Z}} L_j$ is a G_m -bundle. We denote this by Y/μ_X with quotient map $\pi : Y \rightarrow Y/\mu_X$.

Thus we see that every Seifert G_m -bundle can be realized as a (ramified) cover of a G_m -bundle. This observation was a key step in the classification of [OW75].

14 (Compactification of Seifert G_m -bundles). Seifert G_m -bundles can be compactified by adding the missing zero and infinity sections. Adding the zero section corresponds to $\text{Spec}_X \sum_{j \leq 0} L_j$ and adding the infinity section corresponds to $\text{Spec}_X \sum_{j \geq 0} L_j$ in the notation of (10.1). Their union gives a proper morphism $\bar{f} : \bar{Y} \rightarrow X$ which is a \mathbb{P}^1 -bundle over the set where $m(x) = 1$. We have the zero section $X_0 \subset \bar{Y}$ and the infinity section $X_\infty \subset \bar{Y}$. Although \bar{Y} is almost always singular, it is a very useful compactification.

For instance, if X is proper, the zero section $X_0 \subset \bar{Y}$ can be contracted to a point iff $c_1(Y/X)$ is negative. The result of the contraction is the affine variety $\text{Spec}_k \sum_{j \leq 0} H^0(X, L_j)$.

Similarly, if $c_1(Y/X)$ is positive, then the infinity section $X_\infty \subset \bar{Y}$ can be contracted.

Thus Seifert G_m -bundles with $c_1(Y/X)$ negative or positive can all be viewed as a singularity with a good G_m -action (minus the singular point itself). This is the point of view in [Dol75, Pin77, Dem88].

15 (The class group of Seifert bundles). Let $f : Y = Y(L, \sum \frac{b_i}{a_i} D_i) \rightarrow X$ be a Seifert bundle. Our aim is to compute the class group $\text{Cl}(Y)$ of divisors modulo linear equivalence in terms of $\text{Cl}(X)$, L and $\sum \frac{b_i}{a_i} D_i$.

Note first that there is a natural pull back map $f^* : \text{Cl}(X) \rightarrow \text{Cl}(Y)$ since f is equidimensional. Set $D_i^Y := \text{red } f^{-1}(D_i)$.

Proposition 16. [FZ03, 4.22] *Let $f : Y = Y(L, \sum \frac{b_i}{a_i} D_i) \rightarrow X$ be a Seifert G_m -bundle. Then*

$$\text{Cl}(Y) = \langle f^* \text{Cl}(X), D_1^Y, \dots, D_m^Y \rangle / (f^*[D_i] - a_i D_i^Y, f^*[L] + \sum b_i D_i^Y).$$

It is somewhat less precise but more transparent to write this as

$$\text{Cl}(Y) \cong \langle \text{Cl}(X), \frac{1}{a_1}[D_1], \dots, \frac{1}{a_n}[D_n] \rangle / (c_1(Y/X)).$$

Proof. By [FMSS95], if a connected solvable group G acts on a scheme Y then $\text{Cl}(Y)$ is generated by G -invariant divisors and the relations are given by G -eigenfunctions.

The G_m -invariant divisors on Y are the pull backs of divisors on X , except that $f^* D_i$ becomes $a_i D_i^Y$.

A G_m -equivariant rational function on Y can be identified with a G -equivariant rational section of $f_* \mathcal{O}_Y$. Let h be any nonzero rational section of L_1 . Then every G_m -equivariant rational section of $f_* \mathcal{O}_Y$ is of the form $\phi(x)h^k$ for some $k \in \mathbb{Z}$, hence $(h) = 0$ gives all other relations.

Choose h such that it has neither zero nor pole along the divisors D_i . As we computed in (12), h , as a function on Y , has a b_i -fold zero along D_i^Y . This gives the relation $f^*[L] + \sum b_i D_i^Y = 0$. \square

All the rational Chow groups of arbitrary Seifert G -bundles are computed in [Vis89, 4.4].

17 (Seifert G_m^n -bundles). The description of all Seifert G_m^n -bundles for $n \geq 2$ is very similar.

Let X be a normal variety, L_1, \dots, L_n rank 1 reflexive sheaves on X and $\Delta_i = \sum_j s_{ij} D_{ij}$ \mathbb{Q} -divisors such that $[\Delta_i] = 0$. Define

$$\begin{aligned} S(L_1, \dots, L_n, \Delta_1, \dots, \Delta_n) &:= \sum_{j_1, \dots, j_n \in \mathbb{Z}} (L_1^{j_1} \otimes \dots \otimes L_n^{j_n})^{**}([\sum_i j_i \Delta_i]), \quad \text{and} \\ Y(L_1, \dots, L_n, \Delta_1, \dots, \Delta_n) &:= \text{Spec}_X S(L_1, \dots, L_n, \Delta_1, \dots, \Delta_n). \end{aligned}$$

This is a Seifert G_m^n -bundle iff the set of all multi indices (j_1, \dots, j_n) such that

$$(L_1^{j_1} \otimes \dots \otimes L_n^{j_n})^{**}([\sum_i j_i \Delta_i])$$

is locally free and $\sum_i j_i \Delta_i$ is an integral divisor, is a subgroup of rank n of \mathbb{Z}^n .

Exactly as in (7) we obtain that every Seifert G_m^n -bundle $f : Y \rightarrow X$ can be written as $Y \cong Y(L_1, \dots, L_n, \Delta_1, \dots, \Delta_n)$ for some L_i and Δ_i .

2. LOCAL CLASSIFICATION OF SEIFERT G_m -BUNDLES

18 (Quotient description). Let $f : Y(L, \sum s_i D_i) \rightarrow X$ be a Seifert G_m -bundle and $x \in X$ a point. Set $M = m(x)$. By replacing X by a smaller open neighborhood of x if necessary, we may assume that L_M is free and $L_{jM} \cong L_M^{\otimes j}$ for every $j \in \mathbb{Z}$. Fix an isomorphism $\phi : L_M \cong \mathcal{O}_X$.

ϕ defines an algebra structure on $\sum_{j=0}^{M-1} L_j$ by the rules

$$\begin{aligned} L_a \otimes L_b &\xrightarrow{m_{ab}} L_{a+b} \quad \text{if } a+b < M, \text{ and} \\ L_a \otimes L_b &\xrightarrow{m_{ab}} L_{a+b} = L_M \otimes L_{a+b-M} \xrightarrow{\phi^{\otimes 1}} L_{a+b-M} \quad \text{if } a+b \geq M. \end{aligned}$$

We denote this algebra by $S(L, \sum s_i D_i)/(\phi)$, the notation suggesting the natural quotient map

$$\Phi : S(L, \sum s_i D_i) \rightarrow S(L, \sum s_i D_i)/(\phi).$$

Set $Z(L, \sum s_i D_i, \phi) := \text{Spec}_X S(L, \sum s_i D_i)/(\phi)$. It is a Cartier divisor in $Y(L, \sum s_i D_i)$.

It should be noted that $Z(L, \sum s_i D_i, \phi)$ depends only slightly on the choice of ϕ . Any other choice of ϕ can be written as $u \cdot \phi$ where $u \in \mathcal{O}_X$ is invertible near x . If $\text{char } k$ does not divide $m(x)$ then $u = v^{m(x)}$ for some v (at least étale locally) and we obtain that $Z(L, \sum s_i D_i, \phi) \cong Z(L, \sum s_i D_i, u \cdot \phi)$ by the map $L \rightarrow L$ which is multiplication by v .

There is a μ_M -action on $S(L, \sum s_i D_i)/(\phi)$ where L_j is the λ^j -eigensubsheaf.

There is also a natural injection

$$\bar{\Phi} : S(L, \sum s_i D_i) \hookrightarrow S(L, \sum s_i D_i)/(\phi) \otimes_k k[t, t^{-1}]$$

given by $\bar{\Phi}(L_j) = \Phi(L_j) \otimes t^j$.

The μ_M -action on $Z(L, \sum s_i D_i, \phi)$ and the natural μ_M -action on G_m give a diagonal μ_M action on $Z(L, \sum s_i D_i, \phi) \times G_m$ given by

$$L_i \otimes t^j \mapsto \lambda^{i-j} \cdot L_i \otimes t^j.$$

Proposition 19. *The induced map $\bar{\Phi}^* : Z(L, \sum s_i D_i, \phi) \times G_m \rightarrow Y(L, \sum s_i D_i)$ is the quotient map by the above μ_M action. The quotient map $\bar{\Phi}^*$ is étale if $\text{char } k$ does not divide M .*

Proof. $L_i \otimes t^j$ is μ_M -invariant iff $i \equiv j \pmod{M}$. These form exactly the image of $\bar{\Phi}$.

The μ_M action on G_m is fixed point free if $\text{char } k$ does not divide M , hence so is the μ_M action on $Z(L, \sum s_i D_i, \phi) \times G_m$. Thus $\bar{\Phi}^*$ is étale. \square

Proposition 20. *$Y(L, \sum s_i D_i)$ is smooth along $f^{-1}(x)$ iff $Z(L, \sum s_i D_i, \phi)$ is smooth above x .*

Proof. If $\text{char } k$ does not divide $m(x)$ then this follows directly from (19).

To see the general case, note that we have computed in (10) that the reduced fiber Y_x can be identified as $\text{Spec } \sum_{j \in M\mathbb{Z}} k(x)$. Since $S(L, \sum s_i D_i)/(\phi)$ is the sum of terms of degree less than M , we see that $Y_x \subset Y(L, \sum s_i D_i)$ and the hypersurface $Z(L, \sum s_i D_i, \phi) \subset Y(L, \sum s_i D_i)$ intersect scheme theoretically in one point only.

If $Z(L, \sum s_i D_i, \phi)$ is smooth then so is $Y(L, \sum s_i D_i)$ since $Z(L, \sum s_i D_i, \phi)$ is Cartier divisor. Conversely, in a smooth variety, a Cartier divisor transversal to a smooth curve is again smooth. \square

21 (Stacky viewpoint). Instead of considering the geometric quotient Y/G_m as a variety, one can also view it as a stack (see, for instance, [Fan01] for a short introduction). Then we have proved that the $Z(L, \sum s_i D_i, \phi)/\mu_M$ give local charts for the quotient stack. This approach will be especially useful when Y is smooth and $\text{char } k = 0$. In this case the $Z(L, \sum s_i D_i, \phi)$ are smooth and so the quotient stack Y/G_m is an orbifold. This case is studied in detail beginning (25).

While (20) is a smoothness criterion in principle, it is difficult to apply in practice. Our next aim is to develop a smoothness criterion that is truly transparent.

It is easier to work from the other direction. First we recall the local classification of cyclic group actions near a fixed point, then we see how these give actions on a (smooth variety) $\times G_m$. Then we compute the corresponding description as $Y(L, \sum s_i D_i)$.

Notation 22. We consider μ_m -actions on a smooth variety near a fixed point. We always assume that $\text{char } k$ does not divide m .

Etale locally the action is isomorphic to the induced action on the tangent space of the fixed point. Thus we need to classify linear μ_m -actions on \mathbb{A}^n and their quotients. The action of μ_m can be diagonalized and described on $k[z_1, \dots, z_n]$ by $z_i \mapsto \lambda^{a_i} z_i$. We always assume that $\gcd(a_1, \dots, a_n, m) = 1$. The quotient of \mathbb{A}^n by this action of μ_m is denoted by

$$\mathbb{A}^n / \mu_m(a_1, \dots, a_n).$$

(It would be more consistent to denote this by $\mathbb{A}^n / \mu_m(-a_1, \dots, -a_n)$ instead. The two actions have the same invariants, so it is probably not worth while to carry the extra minus sign around.)

This representation is unique, except that we are allowed to permute the a_i and the μ_m action can be given by any generator. That is, if $\gcd(s, m) = 1$ then

$$\mathbb{A}^n / \mu_m(a_1, \dots, a_n) \cong \mathbb{A}^n / \mu_m(sa_1, \dots, sa_n).$$

We write $\mathbb{A}_{\mathbf{z}}^n$ if we need to indicate that we are working with an affine space with coordinates z_i .

Given a_1, \dots, a_n and m set

$$c_i := \gcd(a_1, \dots, \widehat{a_i}, \dots, a_n, m), \quad d_i := a_i c_i / C \quad \text{and} \quad C := \prod c_i. \quad (22.1)$$

Note that the c_i are pairwise relatively prime and C/c_i divides a_i . Observe that $\mu_{c_i} \subset \mu_m$ acts trivially on all but the i th coordinate of \mathbb{A}^n , so it is a quasi reflection.

Therefore the quotient of $\mathbb{A}_{\mathbf{z}}^n$ by $\mu_C \cong \prod \mu_{c_i}$ is again an affine space $\mathbb{A}_{\mathbf{x}}^n$ with $x_i = z_i^{c_i}$. Thus, as a variety,

$$\mathbb{A}_{\mathbf{z}}^n / \mu_m(a_1, \dots, a_n) \cong \mathbb{A}_{\mathbf{x}}^n / \mu_{m/C}(d_1, \dots, d_n).$$

23 (The class group of a quotient singularity). Set $X := \mathbb{A}_{\mathbf{x}}^n / \mu_M(d_1, \dots, d_n)$ where $\gcd(d_1, \dots, \widehat{d_i}, \dots, d_n, M) = 1$ for any i . Define furthermore

$$D_i : (x_i = 0) / \mu_M(d_1, \dots, \widehat{d_i}, \dots, d_n) \subset X,$$

and let

$$\mathcal{O}_X(j) := \epsilon^j\text{-eigenspace of } k[x_1, \dots, x_n],$$

as an \mathcal{O}_X -module. Thus

$$\mathcal{O}_X(j) = \langle \prod x_i^{v_i} : v_i \geq 0, \sum d_i v_i \equiv j \pmod{M} \rangle.$$

Observe that

$$\begin{aligned} \mathcal{O}_X(D_j) &= \langle \prod x_i^{v_i} : v_j \geq -1, v_i \geq 0 \ (i \neq j), \sum d_i v_i \equiv 0 \pmod{M} \rangle \\ &= x_j^{-1} \langle \prod x_i^{w_i} : w_i \geq 0, \sum d_i w_i \equiv d_j \pmod{M} \rangle \cong \mathcal{O}_X(d_j). \end{aligned}$$

Claim 24. The map $j \mapsto \mathcal{O}_X(j)$ gives an isomorphism $\mathbb{Z}/M \cong \text{Cl}(X)$. Its inverse is denoted by c_1 , called the *local Chern class*. (The “natural” Chern class maps $\text{Cl}(X)$ to the group of characters of μ_M . We have identified the latter with \mathbb{Z}/M , but it is probably not a good idea to think of our c_1 as a truly canonical map.)

Proof. This has been known in various forms for a long time; for instance the computations of [Bri68, Mum61] both easily generalize to this case.

It is probably quickest to note that the natural G_m^n -action on \mathbb{A}^n descends to X . By [FMSS95], $\text{Weil}(X)$ is generated by G_m^n -invariant divisors, the D_i , with G_m^n -eigenfunctions giving the relations. \square

25 (Local classification of smooth Seifert bundles). From (19) we see that a Seifert G_m -bundle $f : Y \rightarrow X$ such that Y is smooth along $f^{-1}(x)$ is étale locally isomorphic to a Seifert G_m -bundle of the form

$$f : G_m \times \mathbb{A}^n / \mu_M(r, a_1, \dots, a_n) \rightarrow \mathbb{A}^n / \mu_M(a_1, \dots, a_n),$$

where $r, a_1, \dots, a_n \in \mathbb{Z}/M$, $\gcd(a_1, \dots, a_n, M) = 1$. As before, the Seifert G_m -bundle determines the numbers r, a_1, \dots, a_n modulo M , except that (r, a_1, \dots, a_n) and (sr, sa_1, \dots, sa_n) correspond to the same Seifert G_m -bundle for any s which is relatively prime to M .

The μ_M -action on the coordinate ring of $G_m \times \mathbb{A}^n$ is given by

$$t^s \prod z_i^{u_i} \mapsto \epsilon^{-(rs + \sum a_i u_i)} t^s \prod z_i^{u_i}.$$

Thus the coordinate ring of $G_m \times \mathbb{A}^n / \mu_M(r, a_1, \dots, a_n)$ is

$$k[t^s \prod z_i^{u_i} : s, u_i \geq 0, rs + \sum a_i u_i \equiv 0 \pmod{M}].$$

The G_m -action is

$$t^s \prod z_i^{u_i} \mapsto \lambda^{-s} t^s \prod z_i^{u_i}.$$

In the notation of (10.1), L_j is generated by monomials of the form $t^{-j} \prod z_i^{u_i}$, thus

$$L_j = \langle \prod z_i^{u_i} : u_i \geq 0, \sum a_i u_i \equiv rj \pmod{M} \rangle.$$

Claim 26. The total space $G_m \times \mathbb{A}^n / \mu_M(r, a_1, \dots, a_n)$ is smooth iff either

- (1) r is relatively prime to M , or
- (2) $r = 0$ and $\mathbb{A}^n/\mu_M(a_1, \dots, a_n)$ is smooth.

Proof. If $\text{char } k \nmid M$ then the μ_M -action is base point free if r is relatively prime to M , and so the quotient is smooth. Conversely, if $r \neq 0$, the action of some element of μ_M has a codimension at least 2 fixed point set, so the quotient can not be smooth by the purity of branch loci.

This settles the problem is characteristic 0. The positive characteristic cases can all be lifted to characteristic 0, hence the conditions are necessary.

We still need to show that if r is relatively prime to M then $G_m \times \mathbb{A}^n/\mu_M(r, a_1, \dots, a_n)$ is smooth. Choose e_i such that $re_i \equiv a_i \pmod{M}$. Then the coordinate ring of the quotient can be written as

$$k[x_1 t^{-e_1}, \dots, x_n t^{-e_n}, t^M, t^{-M}]$$

and so $G_m \times \mathbb{A}^n/\mu_M(r, a_1, \dots, a_n) \cong G_m \times \mathbb{A}^n$. \square

We would like to identify L_1 as a reflexive sheaf over $\mathbb{A}^n/\mu_M(a_1, \dots, a_n)$ and also compute $\sum s_i D_i$. We start with some remarks on congruences.

Claim 27. Let c_i and C be as in (22.1).

- (1) One can write r uniquely as $r \equiv lC + \sum a_i b_i \pmod{M}$ where $0 \leq b_i < c_i$ for every i .
- (2) If $\sum a_i u_i \equiv r \pmod{M}$ then $u_j \equiv b_j \pmod{c_j}$ for every j .

Proof. Since $\gcd(a_1, \dots, a_n, M) = 1$, we can write $r \equiv \sum B_i a_i \pmod{M}$. c_i divides every a_j except a_i , so B_i modulo c_i is uniquely determined by $a_i B_i \equiv r \pmod{c_i}$. Let b_i be the remainder of B_i modulo c_i .

Assume that $\sum a_i v_i \equiv lC + \sum a_i b_i \pmod{M}$. Since $c_j | C$ and $c_j | a_i$ for $i \neq j$, we conclude that $a_j v_j \equiv a_j b_j \pmod{c_j}$. Furthermore, a_j is relatively prime to c_j , hence $v_j \equiv b_j \pmod{c_j}$. This proves (2) and the uniqueness part of (1). \square

In order to identify L_1 as module over $\mathbb{A}^n/\mu_M(a_1, \dots, a_n)$ we use (27.1–2) to write it as

$$\begin{aligned} L_1 &= \langle \prod z_i^{u_i} : u_i \geq 0, \sum a_i u_i \equiv r \pmod{M} \rangle \\ &= \langle \prod z_i^{b_i} \rangle \langle \prod z_i^{u_i - b_i} : u_i \geq b_i, \sum a_i u_i \equiv lC + \sum a_i b_i \pmod{M} \rangle \\ &= \langle \prod z_i^{b_i} \rangle \langle \prod x_i^{(u_i - b_i)/c_i} : u_i - b_i \geq 0, \sum a_i c_i \frac{u_i - b_i}{c_i} \equiv lC \pmod{M} \rangle \\ &= \langle \prod z_i^{b_i} \rangle \langle \prod x_i^{v_i} : v_i \geq 0, \sum d_i v_i \equiv l \pmod{M/C} \rangle. \end{aligned}$$

Thus $L_1 \cong \mathcal{O}_X(l)$.

The divisorial part $\sum s_i D_i$ is computed from the map $L_1^{\otimes M} \rightarrow L_M$.

Since $\gcd(d_1, \dots, \widehat{d_j}, \dots, d_n, M/C) = 1$ for every j , the congruence condition $\sum d_i v_i \equiv l \pmod{M/C}$ has solutions where $v_j = 0$. This implies that the M -fold product map

$$\langle \prod x_i^{v_i} : v_i \geq 0, \sum d_i v_i \equiv l \pmod{M/C} \rangle^{\otimes M} \rightarrow \mathcal{O}_X$$

is an isomorphism generically along the D_j , hence its kernel and cokernel are both supported on a codimension 2 set. Thus we conclude that

$$L_1^{[M]} = \left(\prod z_i^{b_i} \right)^M L_M = \left(\prod x_i^{M b_i / c_i} \right) L_M.$$

Thus the divisorial part of the representation of our Seifert G_m -bundle is given by $\frac{1}{M} \sum \frac{M b_i}{c_i} D_i = \sum \frac{b_i}{c_i} D_i$. We can summarize our computations:

Proposition 28. *The Seifert G_m -bundle*

$G_m \times \mathbb{A}_{\mathbf{z}}^n / \mu_M(r, a_1, \dots, a_n) \rightarrow \mathbb{A}^n / \mu_M(a_1, \dots, a_n) \cong \mathbb{A}^n / \mu_{M/C}(d_1, \dots, d_n) = X$
is isomorphic to

$$Y(\mathcal{O}_X(l), \sum \frac{b_i}{c_i} D_i) \rightarrow X,$$

where $c_i := \gcd(a_1, \dots, \widehat{a_i}, \dots, a_n, M)$ and l, b_1, \dots, b_n are the unique solutions to $r \equiv l \prod c_i + \sum a_i b_i \pmod{M}$ satisfying $0 \leq b_i < c_i$ for every i . \square

We are ready to formulate a smoothness criterion that is easy to use in practice.

Proposition 29. *Let $X = \mathbb{A}^n / \mu_M(d_1, \dots, d_n)$ be a quotient singularity such that $\gcd(d_1, \dots, \widehat{d_i}, \dots, d_n, M) = 1$ for every i . Let $D_i = (x_i = 0) / \mu_M(d_1, \dots, \widehat{d_i}, \dots, d_n) \subset X$. Assume that we have integers $0 \leq b_i < c_i$ such that $\gcd(b_i, c_i) = 1$ for every i . The following are equivalent.*

- (1) $Y := Y(\mathcal{O}_X(l), \sum \frac{b_i}{c_i} D_i)$ is smooth.
- (2) (a) $\gcd(c_i, c_j) = 1$ for every $i \neq j$, and
 (b) $\mathcal{O}_X(l \prod c_i)(\sum_i (\prod_{j \neq i} c_j) b_i D_i)$ generates $\text{Cl}(X)$.
- (3) (a) $\gcd(c_i, c_j) = 1$ for every $i \neq j$, and
 (b) $(\prod_i c_i) c_1(Y/X)$ (cf. (9)) is relatively prime to M .

Proof. In the quotient representation $Y = G_m \times \mathbb{A}_{\mathbf{z}}^n / \mu_m(r, a_1, \dots, a_n)$ the c_i are pairwise relatively prime and d_i is relatively prime to c_i . Furthermore, Y is smooth iff $r = (\prod c_i)l + \sum_i (\prod_{j \neq i} c_j) b_i d_i$ is relatively prime to $m = M \prod c_i$.

Pick a prime p dividing c_k . Then $p \nmid b_k d_k$ and p divides every summand of $(\prod c_i)l + \sum_i (\prod_{j \neq i} c_j) b_i d_i$ save $(\prod_{j \neq k} c_j) b_k d_k$. Thus r being relatively prime to $m = M \prod c_i$ is equivalent to being relatively prime to M .

The conditions (b) also imply that $\gcd(c_i, d_i) = 1$ for every i . \square

3. TOPOLOGICAL SEIFERT \mathbb{C}^* -BUNDLES

Definition 30. [OW75] Let X be a normal analytic space and G a real Lie group.

A *topological Seifert G -bundle* over X is a topological space Y together with a G -action and a continuous map $f : Y \rightarrow X$ such that X has an open covering $X = \cup_i U_i$ such that for every i the preimage $f : f^{-1}(U_i) \rightarrow U_i$ is fiber preserving G -equivariantly homeomorphic to a “standard local Seifert G -bundle”

$$f_i : (G \times V_i) / G_i \rightarrow U_i.$$

Here V_i is a normal analytic space, G_i a finite group of biholomorphisms such that $V_i / G_i \cong U_i$ and the G_i -action on $G \times V_i$ is the diagonal action given on G by a homomorphism $\phi_i : G_i \rightarrow G$ composed with the right action of G on itself. The left G -action of G on itself gives the G action on Y .

In order to avoid nontrivial orbifold structures on Y , we always assume that the G_i -action on $G \times V_i$ is fixed point free outside a codimension 2 set.

Every analytic Seifert G -bundle corresponds to a topological Seifert G -bundle by ignoring the analytic structure.

Remark 31. From the topological point of view it is rather unnatural to require that the V_i be complex spaces and it may be interesting to develop a purely topological theory without any analytic assumptions. Our main interest, however, lies in understanding the analytic and topological properties of holomorphic Seifert \mathbb{C}^* -bundles.

Aside from some smoothness questions, the above definition is equivalent to the one in [OW75].

Lemma 32. *Let $f_i : Y_i \rightarrow X$ be two holomorphic Seifert \mathbb{C}^* -bundles which are \mathbb{C}^* -equivariantly homeomorphic. Then there is an open covering $X = \cup U_j$ such that $f_1^{-1}(U_j) \rightarrow U_j$ and $f_2^{-1}(U_j) \rightarrow U_j$ are \mathbb{C}^* -equivariantly biholomorphic for every j .*

Proof. For sufficiently divisible M , consider the quotient Seifert \mathbb{C}^* -bundles $Y_i/\mu_M \rightarrow X$ defined in (13).

Both are \mathbb{C}^* -bundles, hence there is a covering $X = \cup U_j$ such that the $Y_i/\mu_M \rightarrow X$ are trivial over every U_j .

The quotient maps $Y_i \rightarrow Y_i/\mu_M$ are finite degree ramified coverings which are \mathbb{C}^* -equivariantly homeomorphic to each other.

The biholomorphisms $Y_1/\mu_M|_{U_j} \cong Y_2/\mu_M|_{U_j}$ differ from the global homeomorphism $Y_1/\mu_M \sim Y_2/\mu_M$ by a translation by a continuous function $U_j \rightarrow \mathbb{C}^*$. Thus by suitably changing the \mathbb{C}^* -equivariant homeomorphism over U_j we get a commutative diagram

$$\begin{array}{ccc} Y_1|_{U_j} & \sim & Y_2|_{U_j} \\ \downarrow & & \downarrow \\ Y_1/\mu_M|_{U_j} & \cong & Y_2/\mu_M|_{U_j}, \end{array}$$

where \sim is a homeomorphism, \cong a biholomorphism and the vertical arrows are finite degree ramified coverings of analytic spaces. Thus $Y_1|_{U_j}$ is biholomorphic to $Y_2|_{U_j}$ by the uniqueness of analytic structures on finite coverings. \square

The following rather standard result provides an alternate approach to the description of Seifert \mathbb{C}^* -bundles. It also provides an efficient way to compare holomorphic and topological Seifert \mathbb{C}^* -bundles.

Definition 33. Let X be a normal analytic space and G a commutative complex Lie group. Holomorphic sections of $G \times X \rightarrow X$ form a sheaf, denoted by G_X^{an} . Its cohomology groups are denoted by $H^i(X, G^{an})$. Similarly, continuous sections of $G \times X \rightarrow X$ form a sheaf, denoted by G_X^{top} . Its cohomology groups are denoted by $H^i(X, G^{top})$. There are natural “forgetful” maps $H^i(X, G^{an}) \rightarrow H^i(X, G^{top})$.

Note that $(\mathbb{C}^*)_X^{an}$ is usually denoted by \mathcal{O}_X^* .

Proposition 34. *Let X be a normal analytic space and G a commutative complex (resp. real) Lie group. Let $X = \cup U_i$ be an open cover and assume that over each U_i we have an analytic (resp. topological) Seifert G -bundle $Y_i \rightarrow U_i$ and there are G -equivariant biholomorphisms (resp. homeomorphisms) $\phi_{ij} : Y_j|_{U_{ij}} \cong Y_i|_{U_{ij}}$.*

- (1) *There is an obstruction element in the torsion subgroup $H_{tors}^2(X, G^{an})$ (resp. $H_{tors}^2(X, G^{top})$) such that there is a global Seifert G -bundle $Y \rightarrow X$ compatible with these local structures iff the obstruction element is zero.*
- (2) *The set of all such global Seifert G -bundles, up to G -equivariant biholomorphisms (resp. homeomorphisms), is either empty or forms a principal homogeneous space under $H^1(X, G^{an})$ (resp. $H^1(X, G^{top})$).*
- (3) *Both of the constructions commute with the forgetful maps $H^i(X, G^{an}) \rightarrow H^i(X, G^{top})$.*

Proof. The analytic and topological proofs are exactly the same. We write $H^i(X, G)$ to indicate that we can work in either settings.

The isomorphisms ϕ_{ij} can be changed to $\alpha_{ij}\phi_{ij}$ for any $\alpha_{ij} \in H^0(U_{ij}, G)$. These patchings define a global Seifert G -bundle iff

$$\alpha_{ik}\phi_{ik} = \alpha_{ij}\phi_{ij}\alpha_{jk}\phi_{jk} \quad \text{for every } i, j, k.$$

This is equivalent to

$$\alpha_{ij}\alpha_{jk}\alpha_{ki} = (\phi_{ij}\phi_{jk}\phi_{ki})^{-1} \quad \text{for every } i, j, k. \quad (34.4)$$

The products $(\phi_{ij}\phi_{jk}\phi_{ki})^{-1} \in H^0(U_{ijk}, G)$ satisfy the cocycle condition, and they define an element of $H^2(X, G)$, called the obstruction. One can find $\{\alpha_{ij}\}$ satisfying (34.4) iff the obstruction is zero.

Replacing the Y_j by Y_j/μ_M changes the isomorphisms over $U_i \cap U_j$ to ϕ_{ij}^M , hence the obstruction corresponding to the Seifert \mathbb{C}^* -bundles Y_j/μ_M is the M th power of the original obstruction.

If M is sufficiently divisible, the quotients Y_j/μ_M are all \mathbb{C}^* -bundles, and these can always be globalized to the trivial \mathbb{C}^* -bundle. Thus the obstruction is always torsion.

Two choices $\{\alpha_{ij}\}$ and $\{\alpha'_{ij}\}$ give isomorphic Seifert bundles iff there are isomorphisms $\delta_i : Y_i \cong Y_i$ (viewed as elements of $H^0(U_i, G)$) such that

$$\alpha'_{ij}\alpha_{ij}^{-1} = \delta_i\delta_j^{-1}|_{U_{ij}}.$$

Thus $\{\alpha'_{ij}\alpha_{ij}^{-1}\}$ corresponds to a class in $H^1(X, G)$. \square

Example 35. Consider $\mathbb{P}^1 \times \mathbb{P}^1$. Blow up the 4 points all of whose coordinates are in $\{0, \infty\}$ and contract the resulting 4 rational curves of selfintersection -2 . We get a surface S with 4 ordinary double points p_i . The local class group of the points is $\mathbb{Z}/2$.

By looking at all curves on $\mathbb{P}^1 \times \mathbb{P}^1$ we easily see that if $C \subset S$ is any curve and $[C]_i$ is the corresponding element in the local class group of p_i then $\sum_i [C]_i = 0 \in \mathbb{Z}/2$. In particular we obtain that any Seifert \mathbb{C}^* bundle which is trivial on $S \setminus \{p_1\}$ is trivial.

Here $H^3(S, \mathbb{Z}) \cong \mathbb{Z}/2$ and there is a nontrivial obstruction.

Lemma 36. *Notation as above.*

- (1) $H^i(X, (\mathbb{C}^*)^{top}) \cong H^{i+1}(X, \mathbb{Z})$ for $i \geq 1$.
- (2) If $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ then $H^1(X, (\mathbb{C}^*)^{an}) \cong H^2(X, \mathbb{Z})$ and there is an injection $H^2(X, (\mathbb{C}^*)^{an}) \hookrightarrow H^3(X, \mathbb{Z})$.

Proof. Let \mathbb{C}_X^{top} denote the sheaf of continuous complex valued functions. This sheaf is soft and so it has no higher cohomologies (cf. [Bre67, II.9]). Thus the long exact cohomology sequence of the exponential sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C}_X^{top} \xrightarrow{exp} (\mathbb{C}^*)_X^{top} \rightarrow 0$$

proves (1).

The second part follows similarly from the holomorphic exponential sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \rightarrow 0. \quad \square$$

Remark 37. Let X be a normal analytic space with rational singularities such that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$.

Let $f : X' \rightarrow X$ be any resolution of singularities. It is not hard to see (cf. [AM72]) that $\ker \left[H_{tors}^3(X, \mathbb{Z}) \xrightarrow{f^*} H_{tors}^3(X', \mathbb{Z}) \right]$ is independent of the choice of X' .

One can show that the obstructions in (34.1) are in this smaller group. We do not use this in the sequel.

Theorem 38. *Let X be a normal analytic space such that $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$. Then the forgetful map*

$$\left\{ \begin{array}{c} \text{holomorphic} \\ \text{Seifert } \mathbb{C}^*\text{-bundles} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{topological} \\ \text{Seifert } \mathbb{C}^*\text{-bundles} \end{array} \right\}$$

is an isomorphism.

Proof. First we consider injectivity. By (32) if $f_i : Y_i \rightarrow X$ are two holomorphic Seifert \mathbb{C}^* -bundles which are homeomorphic then they are locally biholomorphic to each other.

Thus, by (34) they differ by an element of $H^1(X, (\mathbb{C}^*)^{an})$, which is isomorphic to $H^2(X, \mathbb{Z})$ by (36). Hence they are also different as topological Seifert \mathbb{C}^* -bundles, again by (34) and (36).

In order to prove that every topological Seifert \mathbb{C}^* -bundle has a holomorphic structure, consider the quotient Seifert \mathbb{C}^* -bundle Y/μ_X defined in (13). (Although these were considered in the algebraic case, the construction can be also performed topologically.)

The quotient Y/μ_X is a topological \mathbb{C}^* -bundle, and these are classified by $H^2(X, \mathbb{Z})$. By our assumptions and (36), every topological \mathbb{C}^* -bundle has a unique analytic \mathbb{C}^* -bundle structure.

The quotient map $Y \rightarrow Y/\mu_X$ is a finite degree ramified covering map, and it ramifies along the preimages of the branch divisors of $V_i \rightarrow U_i$ which are analytic subvarieties. Thus Y has a unique analytic structure which makes the quotient map $Y \rightarrow Y/\mu_X$ holomorphic. \square

4. ORBIFOLDS

Definition 39 (Orbifolds). An *orbifold* is a normal variety X covered by étale charts $\cup U_i \rightarrow X$ where each U_i is written as a quotient of a smooth variety by a finite group. That is, for each U_i there is a smooth variety V_i and a finite (reduced) group G_i acting on V_i such that U_i is isomorphic to the quotient space V_i/G_i . The quotient maps are denoted by $\phi_i : V_i \rightarrow U_i$.

We denote an orbifold by X^{orb} .

The compatibility condition between the charts that one needs to assume is that the coordinate projections from the normalization of the fiber products

$$\overline{V_i \times_X V_j} \rightarrow V_i$$

are all étale.

To avoid complications, we only consider the case when the orders $|G_i|$ are not divisible by the characteristic.

Since the transition maps between the charts are étale, the cotangent bundles $\Omega_{V_i}^1$ glue together to the orbifold cotangent bundle $\Omega_{X^{orb}}^1$.

Although we write $X = \cup U_i$, we are always allowed to replace the covering $\{U_i\}$ by a suitable finer covering $\{U_{ij}\}$.

We allow the case when there are codimension 1 fixed point sets. Then the quotient map $\phi_i : V_i \rightarrow U_i$ has branch divisors $D_{ij} \subset U_i$ and ramification divisors

$R_{ij} \subset V_i$. Let m_{ij} denote the ramification index over D_{ij} . Locally near a general point of R_{ij} the map ϕ_i looks like

$$\phi_i : (x_1, x_2, \dots, x_n) \mapsto (z_1 = x_1^{m_{ij}}, z_2 = x_2, \dots, z_n = x_n).$$

The compatibility condition between the charts implies that there are global divisors $D_j \subset X$ and ramification indices m_j such that $D_{ij} = U_i \cap D_j$ and $m_{ij} = m_j$ (after suitable re-indexing).

It will be convenient to codify these data by a single \mathbb{Q} -divisor, called the *branch divisor* of the orbifold,

$$\Delta := \sum (1 - \frac{1}{m_j}) D_j.$$

The orbifold is uniquely determined by the pair (X, Δ) . Indeed, given a point $v \in V_i$, set $u = \phi_i(v) \in U_i$. Then étale locally the quotient map $\phi_i : (v \in V_i) \rightarrow (u \in U_i)$ is uniquely determined by the following condition:

- (1) ϕ is unramified over $U_i \setminus (\text{Sing } U_i \cup \bigcup_j D_j)$.
- (2) The ramification index of ϕ over D_j divides m_j .
- (3) $\phi_i : (v \in V_i) \rightarrow (u \in U_i)$ is the maximal covering satisfying the above 2 conditions.

Slightly inaccurately, we sometimes identify the orbifold with the pair (X, Δ) .

An orbifold (X, Δ) is called *locally cyclic* if all the local charts $U_i \subset X$ are quotients of the V_i by cyclic groups G_i . Thus, étale locally, it can be given by charts

$$\mathbb{A}_{\mathbf{z}}^n / \mu_M(a_1, \dots, a_n),$$

where $\gcd(a_1, \dots, a_n, M) = 1$.

Let $f : Y \rightarrow X$ be a Seifert G_m -bundle and assume that $\text{char } k = 0$. By (20), this defines an orbifold structure on X , denoted by (X, Δ) . Note that the divisors appearing in $\Delta = \sum (1 - \frac{1}{m_j}) D_j$ are the same as the divisors appearing in the representation $Y = Y(L, \sum \frac{b_j}{c_j} D_j)$. Furthermore, $c_j = m_j$ but the orbifold does not carry any information about the b_j or about L .

We write a Seifert G_m -bundle as

$$f : Y \rightarrow (X, \Delta)$$

if we want to emphasize the orbifold structure on the base.

From the point of view of [OW75, Kol04b] it is very natural to choose first (X, Δ) and then consider the effect of various choices of b_i and L later.

Proposition 40. *Let $f : Y \rightarrow (X, \Delta)$ be a Seifert G_m -bundle over a field of characteristic 0, Y smooth. There is an exact sequence*

$$0 \rightarrow f^* \Omega_{X^{\text{orb}}}^1 \rightarrow \Omega_Y^1 \rightarrow \mathcal{O}_Y \rightarrow 0.$$

Proof. For any orbifold chart $U = V/\mu_M$ of (X, Δ) we get an étale chart $G_m \times V/\mu_M$ on Y . Choose coordinates (t, x_1, \dots, x_n) , then

$$\Omega_{G_m \times V}^1 \cong \frac{dt}{t} \mathcal{O}_{G_m \times V} + f^* \Omega_V^1.$$

This decomposition is invariant under the μ_M -action, but the representation of $G_m \times V$ as a direct product is not unique, as we can replace t by $\phi(x)t$ where $\phi \in \mathcal{O}_V$ is any invertible function. Since

$$\frac{d(\phi(x)t)}{\phi(x)t} = \frac{dt}{t} + \sum_i \frac{1}{\phi(x)} \frac{\partial \phi(x)}{\partial x_i} dx_i,$$

we conclude that $\frac{dt}{t}$ gives a well defined global generator of $\Omega_Y^1/f^*\Omega_{X^{orb}}^1$. \square

Corollary 41. [FZ03] *Let $f : Y \rightarrow (X, \Delta)$ be a Seifert G_m -bundle over a field of characteristic 0. Then*

$$K_Y = f^*K_X + \sum (a_i - 1)D_i^Y.$$

Proof. Linear equivalence of divisors is not affected by throwing away a subset of codimension 2, thus we may assume that Y and X are smooth. This case follows directly from (40) by noting that $f^*K_{X^{orb}} = f^*K_X + \sum (a_i - 1)D_i^Y$. \square

This may seem a little unexpected since for the total space $g : Z \rightarrow X$ of a line bundle L the canonical bundle formula is $K_Z = g^*(K_X + L)$. We are, however, looking only at $Y := Z \setminus (\text{zero section})$, and this exactly kills the g^*L part of the formula.

42. Let $f : Y \rightarrow (X, \Delta)$ be a Seifert G_m -bundle over a field of characteristic 0 such that $c_1(Y/X)$ is ample. As noted in (14), this implies that the section at infinity $X_\infty \subset \bar{Y}$ is contractible and we obtain a singularity $0 \in W$ with a good G_m -action.

From (41) and (15) we conclude that K_W is \mathbb{Q} -Cartier iff $K_X + \Delta$ is a rational multiple of $c_1(Y/X)$.

Furthermore, if (X, Δ) is an orbifold, then applying the generalized adjunction formula (as developed in [Kol92, Sec.16]) to $X_\infty \subset \bar{Y}$ we conclude that $0 \in W$ is log terminal (cf. [KM98, 2.34]) iff $-(K_X + \Delta)$ is ample.

5. TOPOLOGY OF SEIFERT \mathbb{C}^* -BUNDLES

In this section the base field is \mathbb{C} .

Our aim is to obtain information about the integral cohomology groups and the fundamental group of a Seifert bundle $f : Y \rightarrow (X, \Delta)$ in terms of (X, Δ) and the Chern class of $Y \rightarrow X$.

The cohomology groups $H^i(Y, \mathbb{Z})$ are computed by a Leray spectral sequence whose E_2 term is

$$E_2^{i,j} = H^i(X, R^j f_* \mathbb{Z}_Y) \Rightarrow H^{i+j}(Y, \mathbb{Z}).$$

Every fiber of f is \mathbb{C}^* , so $R^2 f_* \mathbb{Z}_Y = 0$ and the only interesting higher direct image is $R^1 f_* \mathbb{Z}_Y$. Our first task is to compute this sheaf and its cohomology groups.

Next we consider the simplest edge homomorphism in the spectral sequence

$$\delta : H^0(X, R^1 f_* \mathbb{Z}_Y) \rightarrow H^2(X, \mathbb{Z}),$$

and identify it with the Chern class $m(X)c_1(Y/X)$.

In some cases of interest, these data completely determine the cohomology groups, and even the topology, of Y . Some of these instances are discussed in [OW75, Kol04a].

Proposition 43. *Let $f : Y \rightarrow X$ be a Seifert \mathbb{C}^* -bundle.*

- (1) *There is a natural isomorphism $\tau : R^1 f_* \mathbb{Q}_Y \cong \mathbb{Q}_X$.*
- (2) *There is a natural injection $\tau : R^1 f_* \mathbb{Z}_Y \hookrightarrow \mathbb{Z}_X$ which is an isomorphism over points where $m(x) = 1$.*
- (3) *If $U \subset X$ is connected then*

$$\tau(H^0(U, R^1 f_* \mathbb{Z}_Y)) = m(U) \cdot H^0(U, \mathbb{Z}) \cong m(U) \cdot \mathbb{Z},$$

where $m(U)$ is the lcm of the multiplicities of all fibers over U .

Proof. Pick $x \in X$ and a contractible neighborhood $x \in V \subset X$. Then $f^{-1}(V)$ retracts to $S^1 \subset f^{-1}(x)$ and (together with the orientation) this gives a distinguished generator $\rho \in H^1(f^{-1}(V), \mathbb{Z})$. This in turn determines a cohomology class $\frac{1}{m(x)}\rho \in H^1(f^{-1}(V), \mathbb{Q})$. These normalized cohomology classes are compatible with each other and give a global section of $R^1 f_* \mathbb{Q}_Y$. Thus $R^1 f_* \mathbb{Q}_Y = \mathbb{Q}_X$ and we also obtain the injection $\tau : R^1 f_* \mathbb{Z}_Y \hookrightarrow \mathbb{Z}_X$ as in (2).

If $U \subset X$ is connected, a section $b \in \mathbb{Z} \cong H^0(U, \mathbb{Z}_U)$ is in $\tau(R^1 f_* \mathbb{Z}_Y)$ iff $m(x)$ divides b for every $x \in U$. This is exactly (3). \square

Corollary 44. *The quotient map $\pi : Y \rightarrow Y/\mu_X$ (defined in (13)) induces an isomorphism*

$$H^0(X, R^1 f_* \mathbb{Z}_Y) \cong \pi^* H^0(X, R^1 (f/\mu_X)_* \mathbb{Z}_{Y/\mu_X}).$$

Under this isomorphism, the edge homomorphism

$$\delta : H^0(X, R^1 f_* \mathbb{Q}_Y) \rightarrow H^2(X, \mathbb{Q})$$

is identified with the Chern class $c_1(Y/X)$. Thus the image of

$$\delta : H^0(X, R^1 f_* \mathbb{Z}_Y) \rightarrow H^2(X, \mathbb{Z})$$

is generated by $c_1(Y/\mu_X) = m(X)c_1(Y/X)$.

Proof. If $m(X)$ is the lcm of the multiplicities of all fibers then

$$\tau(\pi^* H^0(X, R^1 (f/\mu_X)_* \mathbb{Z}_{Y/\mu_X})) = m(X) \cdot H^0(X, \mathbb{Z})$$

and so it agrees with $\tau(H^0(X, R^1 f_* \mathbb{Z}_Y))$.

The map π induces a map between the spectral sequences

$$H^i(X, R^j f_* \mathbb{Z}_{Y/\mu_X}) \Rightarrow H^{i+j}(Y/\mu_X, \mathbb{Z}) \quad \text{and} \quad H^i(X, R^j f_* \mathbb{Z}_Y) \Rightarrow H^{i+j}(Y, \mathbb{Z}),$$

thus δ is identified with the corresponding edge homomorphism

$$\delta : H^0(X, R^1 (f/\mu_X)_* \mathbb{Z}_{Y/\mu_X}) \rightarrow H^2(X, \mathbb{Z}).$$

For a \mathbb{C}^* -bundle this is exactly the Chern class of Y/μ_X . \square

Corollary 45. *The quotient map $\pi : Y \rightarrow Y/\mu_X$ induces an isomorphism on rational cohomologies. Thus Y is a rational homology sphere iff*

$$H^*(X, \mathbb{Q}) \cong \mathbb{Q}[c_1(Y/X)]/(c_1(Y/X)^{n+1}) \quad \text{where } n = \dim X.$$

Proof. We have seen that π induces an isomorphism between the spectral sequences with \mathbb{Q} -coefficients, giving the first claim. A circle bundle $M \rightarrow X$ is an integral (resp. rational) homology sphere iff $c_1(M/X)$ generates $H^*(X, \mathbb{Z})$ (resp. $H^*(X, \mathbb{Q})$). \square

Note 46. The only simply connected smooth projective variety such that $H^*(X, \mathbb{Z})$ is generated by a degree 2 element that I know is \mathbb{P}^n . All Seifert bundles $Y \rightarrow \mathbb{P}^n$ homeomorphic to S^{2n+1} were described in [OW75].

On the other hand, there are many other Seifert bundles over singular varieties $Y \rightarrow X$ such that Y is homeomorphic to S^{2n+1} . These give interesting examples of Einstein metrics on S^{2n+1} , see [BGK03].

We see that (43) describes the sheaf $R^1 f_* \mathbb{Z}_Y$ completely in terms of (X, Δ) , but it is not always easy to compute its cohomologies based on this description. There are, however, some cases where this is quite straightforward.

Proposition 47. *Let $f : Y \rightarrow (X, \sum(1 - \frac{1}{m_i})D_i)$ be a Seifert bundle and assume that X is smooth. Then there is an exact sequence*

$$0 \rightarrow R^1 f_* \mathbb{Z}_Y \xrightarrow{\tau} \mathbb{Z}_X \rightarrow \sum_i \mathbb{Z}_{D_i}/m_i \rightarrow 0.$$

Proof. Note that m_i and m_j are relatively prime if $D_i \cap D_j \neq \emptyset$. It is now clear that the kernel of $\mathbb{Z}_X \rightarrow \sum_i \mathbb{Z}_{D_i}/m_i$ has the same sections as described in (43.3). \square

We get a slightly more complicated description if X has only isolated singularities.

Proposition 48. *Let $f : Y \rightarrow (X, \sum(1 - \frac{1}{m_i})D_i)$ be a Seifert bundle. Assume that X has only isolated singularities of type \mathbb{A}^n/μ_{n_j} at $P_j \in X$ as in (22). Then there is an exact sequence*

$$0 \rightarrow R^1 f_* \mathbb{Z}_Y \xrightarrow{\tau} \mathbb{Z}_X \rightarrow Q \rightarrow 0.$$

and in turn Q sits in another exact sequence

$$0 \rightarrow \sum_j \mathbb{Z}_{P_j}/n_j \rightarrow Q \rightarrow \sum_i \mathbb{Z}_{D_i}/m_i \rightarrow 0. \quad \square$$

Definition 49. Let $(X, \sum_i(1 - \frac{1}{m_i})D_i)$ be an orbifold and $X^0 \subset X$ the smooth locus of X . The orbifold fundamental group $\pi_1^{orb}(X, \Delta)$ is the fundamental group of $X^0 \setminus \text{Supp } \Delta$ modulo the relations: if γ is any small loop around D_i at a general point then $\gamma^{m_i} = 1$.

Note that $\pi_1^{orb}(X, 0)$ is usually different from $\pi_1(X)$.

The abelianization of $\pi_1^{orb}(X^0, \Delta)$ is denoted by $H_1^{orb}(X^0, \Delta)$, called the *abelian orbifold fundamental group*. (I do not know how to define analogous orbifold homology groups H_i^{orb} for $i \geq 2$.)

The following is a straightforward generalization of the computation of the fundamental group of 3-dimensional Seifert bundles, cf. [Sei32].

Proposition 50. *Let $f : Y \rightarrow (X, \Delta)$ be a Seifert bundle and let $X^0 \subset X$ denote the smooth locus. Assume that Y is smooth. There is an exact sequence*

$$\pi_1(S^1) \rightarrow \pi_1(Y) \rightarrow \pi_1^{orb}(X, \Delta) \rightarrow 1.$$

Proof. Removing a real codimension 4 set $f^{-1}(X \setminus X^0)$ from Y does not change the fundamental group, thus we may assume that X is smooth.

We have a surjection

$$f_* : \pi_1(f^{-1}(X \setminus \text{Supp } \Delta)) \twoheadrightarrow \pi_1(X \setminus \text{Supp } \Delta).$$

Let γ_i be a small loop around D_i at a general point. Then γ_i lifts to a loop around $f^{-1}(D_i)$ and the lifting of $\gamma_i^{m_i}$ contracts in Y . Thus f_* descends to a surjection

$$f_* : \pi_1(Y) \twoheadrightarrow \pi_1^{orb}(X, \Delta).$$

In order to compute the kernel, we can pass to the universal orbifold cover of (X, Δ) and so we assume that $\pi_1^{orb}(X, \Delta) = 1$.

Let $\gamma \in Y$ be a closed loop. We can perturb it and assume that it is contained in $Y \setminus f^{-1}(\text{Supp } \Delta)$. The image $f(\gamma) \in X \setminus \text{Supp } \Delta$ need not be contractible, but it is in the normal subgroup generated by the above $\gamma_i^{m_i}$. The latter are all images of loops that are contractible in Y , thus by composing γ with these we may assume

that $f(\gamma)$ is contractible in $X \setminus \text{Supp } \Delta$. Since $Y \setminus f^{-1}(\text{Supp } \Delta) \rightarrow X \setminus \text{Supp } \Delta$ is a circle bundle, γ is thus homotopic to a multiple of the fiber. \square

51. The determination of $\pi_1^{\text{orb}}(X, \Delta)$ seems rather tricky in general, but its abelianization is fully computable. The case when $H_1(X^0, \mathbb{Z}) = 0$ is especially easy to state.

Proposition 52. [OW75, 4.6] *Let X be a complex manifold such that $H_1(X, \mathbb{Z}) = 0$ and let $D_1, \dots, D_n \subset X$ be smooth divisors intersecting transversally.*

- (1) $H_1^{\text{orb}}(X, \sum (1 - \frac{1}{c_i}) D_i)$ is given by generators g_1, \dots, g_n and relations
 - (a) $c_i g_i = 0$ for $i = 1, \dots, n$, and
 - (b) $\sum g_i ([D_i] \cap \eta) = 0$ for every $\eta \in H_2(X, \mathbb{Z})$.
- (2) For any line bundle L , $H_1(Y(L, \sum \frac{b_i}{c_i} D_i), \mathbb{Z})$ is given by generators k, g_1, \dots, g_n and relations
 - (a) $c_i g_i + b_i k = 0$ for $i = 1, \dots, n$, and
 - (b) $k(c_1(L) \cap \eta) - \sum g_i ([D_i] \cap \eta) = 0$ for every $\eta \in H_2(X, \mathbb{Z})$. \square

Proposition 53. *Yet (X, Δ) be an orbifold and assume that $H_1^{\text{orb}}(X, \Delta) = 0$. Then a Seifert \mathbb{C}^* -bundle $f : Y \rightarrow (X, \Delta)$ is uniquely determined by its Chern class $c_1(Y/X) \in H^2(X, \mathbb{Q})$.*

Proof. Let $Y^j \rightarrow (X, \Delta)$ be two Seifert \mathbb{C}^* -bundles constructed from the invariants $(\sum (b_i^j/c_i) D_i, L^j)$ as in (7). The equality of their Chern classes means that

$$c_1(L^1) + \sum \frac{b_i^1}{c_i} [D_i] = c_1(L^2) + \sum \frac{b_i^2}{c_i} [D_i] \in H^2(X, \mathbb{Q}).$$

Let $X^0 \subset X$ denote the smooth locus and set $M := \text{lcm}\{c_i\}$. Since $H_1(X^0, \mathbb{Z}) = 0$, there is no torsion in $H^2(X^0, \mathbb{Z})$, and so after restricting to X^0 we get an equality in integral cohomology

$$M \cdot (c_1(L^1) - c_1(L^2)) + \sum (b_i^1 - b_i^2) \frac{M}{c_i} [D_i] = 0 \in H^2(X^0, \mathbb{Z}). \quad (53.1)$$

$H_1(X, \mathbb{Z}) = 0$ also implies that $H^1(X, \mathcal{O}_X) = 0$, hence the Picard group of X^0 injects into $H^2(X^0, \mathbb{Z})$, and so (53.1) is also a linear equivalence.

Set $\beta_i = b_i^1 - b_i^2$ if $b_i^1 - b_i^2 \geq 0$ and $\beta_i = c_i + b_i^1 - b_i^2$ if $b_i^1 - b_i^2 < 0$. Then (53.1) can be rearranged to

$$\sum \beta_i \frac{M}{c_i} [D_i] = M \cdot c_1(N),$$

where N is some line bundle on X^0 and $0 \leq \beta_i < c_i$. This corresponds to an M -sheeted covering of X^0 which ramifies along D_i with ramification index $= c_i / \gcd(\beta_i, c_i)$. This covering represents a nontrivial element of $H_1^{\text{orb}}(X, \Delta)$, unless $c_i | \beta_i$ for every i . This is only possible if $\beta_i = 0$ for every i . Thus $b_i^1 = b_i^2$ for every i and hence also $L^1 \cong L^2$. \square

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Princeton University, Princeton NJ 08544-1000

`kollar@math.princeton.edu`